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ORIGINAL RESEARCH

Portfolio revision under mean-variance and mean-CVaR with transaction costs

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Abstract The portfolio revision process usually begins with a portfolio of assets rather than cash. As a result, some assets must be liquidated to permit investment in other assets, incurring transaction costs that should be directly integrated into the portfolio optimization problem. This paper discusses and analyzes the impact of transaction costs on the optimal portfolio under mean-variance and mean-conditional value-at-risk strategies. In addition, we present some analytical solutions and empirical evidence for some special situations to understand the impact of transaction costs on the portfolio revision process.

Keywords Portfolio revision · Transaction costs · Mean-variance · Conditional value-at-risk (CVaR)

JEL Classification G11 · C61

1 Introduction

In Markowitz's (1952) article, as well as in his book published 7 years later (Markowitz 1959), he suggests that investors should decide the allocation of their investment on the basis of a trade-off between return and risk. More specifically, Markowitz quantifies the "return" and "risk" of a portfolio as the mean and variance of the random portfolio return, and shows that one portfolio is preferred to another one if and only if it has higher expected

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return and lower risk. This approach to portfolio selection allows for convenient computational recipes and geometric interpretations of the trade-off.[1] The mean-variance model is so intuitive and so strong that it has been continually applied to different areas within finance and risk management (see Fabozzi et al. 2002 for applications).

Considering that investment decisions are usually made starting with a portfolio rather than cash, and consequently some assets must be liquidated to permit investment in others, Chen et al. (1971) were among the first to address this problem by considering the transaction costs incurred in the process of portfolio revision. Actually, the costs associated with buying or selling an earning asset (hereafter called transaction costs) should be directly integrated into the portfolio optimization problem.

Since Chen et al. (1971), stimulating developments have been proposed for portfolio revision with a variety of transaction costs. Here we mention a few important contributions in the past two decades. Davis and Norman (1990) study the consumption and investment decision in the case where the optimal buying and selling policies are charged equal transaction costs. Adcock and Meade (1994) add a linear term for transaction costs to the mean-variance risk term and minimize this quantity. Konno and Wijayanayake (2001) consider a cost structure that is concave in the mean-absolute deviation model. Finally, Lobo et al. (2007) provide a model for the case of linear and fixed transaction costs that maximizes the expected return subject to the variance constraint.

Michaud (1998) finds two interesting phenomena. First, the mean-variance model usually generates high portfolio turnover rates. Second, the optimal portfolios are often counter-intuitive because they are characterized by a lack of diversification with extreme allocations made to just a few assets. For these reasons, it seems interesting to examine how transaction costs in the mean-risk model affect portfolio turnover and diversification. That is, do transaction costs matter in portfolio revision, and could the control of transaction costs improve portfolio performance?

Although the expected return can be easily quantified, there is no consensus on the measurement of risk in the mean-risk framework. That is, without a set of unified axioms, we cannot deduce if "the best" risk measure exists. The concept of a coherent risk measure as introduced by Artzner et al. (1999) provides guidance for identifying the properties of a good risk measure. Several alternative risk measures for portfolio optimization have been proposed in the literature. These portfolio risk measures fall into two classes of risk measures: dispersion measures and safety-first measures. Several of these proposed risk measures have become widely utilized in practice.[2] Among those alternative risk measures, by far the two most popular portfolio risk measures are value at risk (VaR) and conditional value-at-risk (CVaR). Although VaR and CVaR are superior to variance because they provide better measures of downside risk than variance and allow for asymmetric distributions, there are two reasons for the preference in portfolio and risk management of CVaR relative to VaR.[3] First, CVaR takes into account not only the probability but also the size of the loss. Second, unlike VaR, CVaR satisfies the properties for a coherent risk measure while VaR fails to do so.

[1] Although Elton et al. (1976) argue that the Markowitz model is needlessly formalistic and introduce less computationally intensive alternative models of portfolio selection, Frankfurter et al. (1999) find that the ex post performance of the mean-variance model outperforms these alternative models.

[2] For a summary of these measures, see Ortobelli et al. (2005). The use of alternative risk measures in practice is described in Dembo and Rosen (2000) and Ortobelli et al. (2005).

[3] Baixauli and Alvarez (2006) examine the impact of different conditional distribution functions on the accuracy of VaR estimates.

In this paper, we revisit the portfolio revision problem with transaction costs using variance and CVaR as risk measures. The reason for this choice is natural: These two risk measures are typical representatives of symmetric and asymmetric risk measures. Based on the empirical evidence, although the monthly or longer returns of financial assets might exhibit the property of a normal distribution, weekly or shorter (especially daily) returns exhibit skewness and kurtosis, calling for an asymmetric risk measure. The variance and CVaR can accommodate different kinds of risky assets and related returns. Interest in the mean-variance model and in the mean-CVaR model is apparent in the recent literature (Agarwal and Naik 2004; Levy and Levy 2004; Alexander et al. 2007).

We focus on static portfolio revision (single period), i.e., on revision based on a single decision at the start of the horizon, although the revision may cover several time periods (multiple periods). The main reasons for doing so are that the current multi-period portfolio strategies—such as multi-stage mean-variance criteria—are employed in a myopic manner and the decision maker in each period maximizes the next-period objective. While this strategy allows analytical tractability and abstracts from dynamic revising considerations, there is growing evidence that temporal portfolio revision may compromise a significant part of the total demand for risky assets (Campbell and Viceira 1999; Brandt 1999). Actually, the key issue has been the inability to directly apply the "principle of optimality" of the traditional dynamic programming approach due to the failure of the iterated expectations property for the mean-variance or mean-CVaR objectives.

This paper differs from Chen et al. (2010) in three ways. First, in this paper we explore how transaction costs affect portfolio revision when a downside risk measure such as CVaR is used. Second, we propose how to integrate CVaR and variance to deal with the difficulty of estimating the expected return, hence the difficulty of finding a robust portfolio. Third, the empirical evidence on the effect of transaction costs on mean-variance and mean-CVaR is different. In particular, we compare the efficient frontiers in the mean-deviation and mean-CVaR spaces and demonstrate that although their efficient frontiers are close to each other when ignoring transaction costs, they are significantly different in the presence of transaction costs.

Black and Litterman (1992), as well as others (Best and Grauer 1991; Chopra 1993; Chopra and Ziemba 1993), have pointed out that for the mean-variance framework, a small perturbation of the inputs may result in a large change in the optimal portfolio, implying that the parameters should be estimated precisely enough, especially for the expected return. However, the expected returns are estimated from real data, which are often prone to errors. Thus, the lack of robustness of inputs usually entails extreme positions in the assets of the optimal portfolio and delivers a poor out-of-sample performance. To reduce the uncertainty in the expected return, we also investigate the effect of transaction costs with the CVaR-variance model recently proposed by Zhu et al. (2007).[4]

The rest of this paper proceeds as follows. Section 2 describes the basic formulation for the portfolio revision problem with transaction costs. Section 3 considers the portfolio revision decision of Chen et al. (1971) in the mean-variance framework. Section 4 discusses portfolio revision using a CVaR objective. Section 5 considers portfolio revision with a CVaR-variance model. Section 6 gives a simple practical application, followed by a summary of our conclusions in Sect. 7.

[4] For a comprehensive discussion of robust portfolio management and the associated solution methods, see Fabozzi et al. (2007).

2 Problem formulation

In this section, we formulate the portfolio revision problem with transaction costs as a standard mathematical optimization problem with the mean-risk framework. We will employ the following notation.

N:	number of risky assets;
e:	vector with all entries equal to ones;
y^0:	amount invested in the risk-free asset before portfolio revision;
y:	amount invested in the risk-free asset after portfolio revision;
x^0:	initial risky assets before portfolio revision;
x^b:	purchases of the risky assets;
x^s:	sales of the risky assets;
x:	portfolio invested in risky assets after portfolio revision, $x = x^0 + x^b - x^s$;
$c^b(\cdot)$:	transaction costs associated with buying risky assets where $c^b(\cdot) \geq 0$ (for cash we set $c_{\text{cash}} = 0$, that is, one only pays for buying and selling the assets and not for moving the cash in and out of the account);
$c^s(\cdot)$:	transaction costs associated with selling risky assets where $c^s(\cdot) \geq 0$;
r_f :	risk-free rate of interest;
r :	actual return of risky assets with expectation $\bar{r} = \mathbb{E}(r)$ and $\bar{r} \neq r_f e$.
$\Sigma\,(\succ 0)$:	covariance matrix of the portfolio return.

Although the concept of the desirability of diversification can be traced back to Daniel Bernoulli who argues by example that it is advisable for a risk-averse investor to divide goods which are exposed to some small danger into several portions rather than to risk them all together, Markowitz is the first one to mathematically formulate the idea of the diversification of an investment portfolio by defining the variance as a measure of economic risk (Markowitz 1999; Rubinstein 2002). Through diversification, risk can be reduced (but not necessarily eliminated) without changing expected portfolio return. Markowitz rejects the traditional hypothesis that an investor should simply maximize expected returns. Instead, he suggests that an investor should maximize expected portfolio return while minimizing portfolio risk of return, implying a trade-off between expected return and risk.

While the mean-risk model was first proposed for the portfolio optimization problem where the economic conditions and investment opportunities are assumed to be static over the planned horizon, the composition of a portfolio of risky assets, however, will generally change over time because of random outcomes of the returns on its constituent assets in the subperiods prior to the horizon. Adjustment of the proportions of the assets may thus be necessary in order to re-establish an efficient portfolio at the beginning of each sub-period in the planned interval. The investor would also want to adjust the portfolio composition if his expectations or risk aversion changed. The opportunities to adjust the portfolio enable the investor to increase his expected utility at the horizon and therefore should be taken into account in making investment decisions. Given that the investment decisions are usually made starting with a portfolio of assets rather than cash, some assets must be liquidated to permit investment in other assets, incurring transaction costs in the process of portfolio revision.

More specifically, consider an investment portfolio that consists of holdings in some or all of n risky assets and one risk-free asset. Suppose the expected return, risk, and transaction costs are $r(x, y)$, $\rho(x)$, and $c(x)$, respectively. We can easily formulate the problem as

$$(P_0) \qquad \max \quad r(x, y) - \gamma\rho(x) \tag{1}$$

$$\text{s.t.} \quad x = x^0 + x^b - x^s, \tag{2}$$

$$y + e'x + c(x) \leq 1, \tag{3}$$

$$x^b \cdot x^s = 0, \tag{4}$$

$$x^b \geq 0, \, x^s \geq 0, \tag{5}$$

where $x^b \cdot x^s = 0$ is a complementary constraint, i.e., $x_j^b x_j^s = 0$ for $j = 1, \cdots, n$. $\gamma > 0$ is the coefficient of risk aversion (the larger the value of γ, the more reluctant the investor is to take on risk in exchange for expected return). Notice here that we do not include the position y in $\rho(x)$ and $c(x)$ since we have assumed that increasing or decreasing the risk-free asset does not incur any risk and transaction costs.

The number x in (2) represents the portfolio position to be chosen explicitly through sales x^s and purchase x^b that are adjustments to the initial holding x^0. The second constraint (3) is the budget constraint. In contrast to the traditional constraint, there is a new term, transaction costs. Without loss of generality, we normalize the investor's initial wealth, i.e., $y^0 + e' x^0 = 1$. The complementary constraint (4) and the nonnegative constraint (5) rule out the possibility of simultaneous purchases and sales. In practice, simultaneously buying and selling (choosing $x^b > 0$ and $x^s > 0$) can never be optimal because making the allocation to one asset increases and decreases at the same time, thereby incurring unnecessary transaction costs (Dybvig 2005).

While we leave the specification of portfolio risk $\rho(x)$ to later, throughout this paper we assume that transaction costs are separable; that is,

$$c^b(x) = \sum_{j=1}^{n} c_j^b(x_j),$$

$$c^s(x) = \sum_{j=1}^{n} c_j^s(x_j),$$

where $c_j(\cdot)$ is the transaction cost function for asset j. We will focus on the proportional transaction cost—proportional to the total dollar value of the selling/buying assets—and investigate its impact on the portfolio revision policy.[5] For the treatment of non-convex transaction costs, see Konno and Wijayanayake (2001) and Best et al. (2005). Hence, a balance constraint that maintains the self-refinancing strategy including transaction costs is given as

$$y + e'x + \sum_{j=1}^{n} c_j^b x_j^b + \sum_{j=1}^{n} c_j^s x_j^s \leq e'x^0 + y^0 = 1.$$

For computational convenience, some papers directly discard the complementary condition (4) (see Krokhmal et al. 2001; Lobo et al. 2007 for instance). Theoretically,

[5] One might argue that because transaction costs are measured in the same dollars as the mean return, it should simply be added to the mean. However, this is incorrect. Transaction costs affect both the expected value and the variance of the ending wealth. Moreover, transaction costs cannot be directly added to the mean. The reason is that given a portfolio, the objective is to rebalance it for a new optimal one. But it may be possible that the initial weight for one asset is optimal in the new portfolio. In this case, it is unnecessary to change that asset's weight in the portfolio and so it does not make sense to add the transaction cost to the mean.

discarding the complementary condition may lead to buying and selling a particular asset simultaneously, which is obviously unreasonable. Also, if the risk-aversion coefficient γ is very large, then we may always select an optimal portfolio with zero weights on the risky asset ($x = 0$), hence zero risk ($\rho(0) = 0$) by buying and selling the assets or investing all wealth in the risk-free asset. However, Mitchell and Braun (2004) prove that the intractable complementary constraint (4) can be removed in the presence of a risk-free asset.

In summary, we can equivalently rewrite (P_0) as

$$(P_1) \qquad \max \quad r_f y + \bar{r}' x - \gamma \rho(x), \tag{6}$$

$$\text{s.t.} \quad x = x^0 + x^b - x^s, \tag{7}$$

$$y + e'x + \sum_{j=1}^{n} c_j^b x_j^b + \sum_{j=1}^{n} c_j^s x_j^s \leq 1, \tag{8}$$

$$x^b \cdot x^s = 0, \tag{9}$$

$$x^b \geq 0, \quad x^s \geq 0. \tag{10}$$

Note that (P_1) usually has some equivalent counterparts in the sense of expected return-risk-cost efficient frontier: (1) maximizing expected return subject to given risk and transaction costs or (2) minimizing risk subject to a given expected return or transaction costs. Although maximizing expected return subject to a given risk and transaction costs may be especially appealing to practitioners who have trouble quantifying their preferences but may have an idea how much volatility and transaction costs are acceptable, we will only focus on problem (P_1) and analyze the efficient frontier by varying the risk-aversion parameter γ and transaction costs.

3 Mean-variance framework

Following Chen et al. (1971), we summarize the portfolio revision problem in the mean-variance framework as

$$(P_2) \qquad \max \quad (r_f y + \bar{r}' x) - \gamma x' \Sigma x \tag{11}$$

$$\text{s.t.} \quad x = x^0 + x^b - x^s, \tag{12}$$

$$y + e'x + c(x) \leq 1, \tag{13}$$

$$x^b \geq 0, \ x^s \geq 0. \tag{14}$$

In what follows, we investigate two special cases where a closed-form solution can be obtained from (P_2), and then explore the impact of transaction costs on the optimal portfolio position (Chen et al. 1971).

3.1 Analytical results in the case of one risky asset and one risk-free asset

Suppose there are two assets in the portfolio: one risky asset and one risk-free asset with the initial amount denoted by x^0 and y^0 ($x^0 + y^0 = 1$), respectively. The risky asset has mean $\bar{r}$ and variance σ^2, whereas the risk-free asset has constant return rate r_f and variance zero. As analyzed above, it is never optimal to have $x^b > 0$ and $x^s > 0$ at the same time because that would incur both c^b and c^s on the round-trip. We assume the optimal strategy calls for buying some risky asset, i.e., $x^b > 0$ and $x^s = 0$. In this case, a risk-averse investor's strategy from (P_2) reduces to

$$\begin{aligned} \max \quad & r_f y + \bar{r}' x - \gamma \sigma^2 x^2 \\ \text{s.t.} \quad & x = x^0 + x^b, \\ & y + x + x^b c^b = 1. \end{aligned}$$

With the first-order condition, we have the optimal solution

$$x^* = x^0 + x^{b*} = \frac{\bar{r} - r_f(1 + c^b)}{2\gamma\sigma^2} = \frac{\bar{r} - r_f}{2\gamma\sigma^2} - \frac{r_f c^b}{2\gamma\sigma^2}.$$

Obviously, in the buying case, there are two terms. The first term is the optimal position (with the highest Sharpe ratio), $\frac{\bar{r}-r_f}{2\gamma\sigma^2}$, without considering the transaction costs. The second term is the amount resulting from the transaction costs for buying the risky asset.

Now we consider the contrary case where the optimal strategy is to sell some risky asset, i.e., $x^b = 0$ and $x^s > 0$. Then the portfolio optimization problem (P_2) reduces to

$$\begin{aligned} \max \quad & r_f y + \bar{r} x - \gamma \sigma^2 x^2 \\ \text{s.t.} \quad & x = x^0 - x^s, \\ & y + x + x^s c^s = 1, \end{aligned}$$

which delivers the optimal position as

$$x^* = x^0 - x^{s*} = \frac{\bar{r} - r_f(1 - c^s)}{2\gamma\sigma^2} = \frac{\bar{r} - r_f}{2\gamma\sigma^2} + \frac{r_f c^s}{2\gamma\sigma^2}.$$

Similar to the buying case, the optimal solution also has two terms: the first one is the optimal solution without considering transaction costs, and the second is the impact of transaction costs incurred from portfolio revision.[6] Also, if $\bar{r}$ is fixed, $c^b > 0$ and $c^s > 0$, it is apparent that

$$\frac{\bar{r} - r_f}{2\gamma\sigma^2} + \frac{r_f c^s}{2\gamma\sigma^2} > \frac{\bar{r} - r_f}{2\gamma\sigma^2} - \frac{r_f c^b}{2\gamma\sigma^2},$$

which implies that if the initial position x^0 is in $\left[\frac{\bar{r}-r_f}{2\gamma\sigma^2} - \frac{r_f c^b}{2\gamma\sigma^2}, \frac{\bar{r}-r_f}{2\gamma\sigma^2} + \frac{r_f c^s}{2\gamma\sigma^2}\right]$, there is no portfolio revision, no trade region.

It should be mentioned that we may generalize and expand the risky asset as the market portfolio (or the market index) which has the highest Sharpe ratio. In this case, all investors with different risk aversion γ hold a mix of the market portfolio and the risk-free asset according to the two-fund separation theorem. If the transaction cost c^b or c^s is too large, then there is no portfolio revision (see Dybvig 2005).

3.2 Analytical results in the case of two risky assets

Now we consider the case of two risky assets with random returns r_1 and r_2, with means of $\bar{r}_1$ and $\bar{r}_2$, and variances of σ_1^2 and σ_2^2. The correlation between these two assets is assumed

[6] Both the buying and selling cases are consistent with footnote 5 that transaction costs cannot be simply added to the mean.

to be ρ. The composition of the optimal portfolio is the solution of the maximization problem below, assuming symmetric transaction costs, $c_j^b(x) = c^b(x)$ and $c_j^s(x) = c^s|x|$; and further, the optimal strategy calls for buying some asset 1 and selling some asset 2, i.e., x_1^b, $x_2^s > 0$:

$$\begin{aligned} \max \quad & (\bar{r}_1 x_1 + \bar{r}_2 x_2) - \gamma(x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2\gamma\rho\sigma_1\sigma_2 x_1 x_2) \\ \text{s.t.} \quad & x_1 = x_1^b + x_1^0,\ x_2 = x_2^0 - x_2^s, \\ & x_1 + x_2 + x_1^b c + x_2^s c = 1. \end{aligned} \tag{15}$$

Solving (15) yields the following solution:

$$x_1^* = x_1^0 + x_1^{b*} = \frac{\sigma_2[\bar{r}_1 - \lambda(1+c^b)] - \rho\sigma_1[\bar{r}_2 - \lambda(1-c^s)]}{2\gamma\sigma_1^2\sigma_2(1-\rho^2)},$$
$$x_2^* = x_2^0 - x_1^{s*} = \frac{\sigma_1[\bar{r}_2 - \lambda(1-c^s)] - \rho\sigma_2[\bar{r}_1 - \lambda(1+c^b)]}{2\gamma\sigma_1\sigma_2^2(1-\rho^2)},$$

where λ is the Lagrange multiplier and can be easily calculated from the budget constraint.

Strictly speaking, we should also develop the reverse case, namely, to sell asset 1 and buy asset 2. The solution is entirely symmetric to (x_1^*,x_2^*) and is found simply by interchanging subscripts. In addition, there is also a region where the investor will not revise the portfolio at all; that is, $x_1^* = x_1^0$ and $x_2^* = x_2^0$.

4 Mean-CVaR framework

In this section, we consider the mean-CVaR model for portfolio revision with transaction costs. First, let $f(x, r)$ denote the loss of a portfolio with decision vector $x \in X \subseteq \Re^N$ and random vector $r \in \Re^N$ that represents the actual portfolio return. Suppose $\mathbb{E}(|f(x,r)|) < +\infty$ for each $x \in \Re^N$ and r has a continuous density function $p(r)$.[7] For the purpose of clarity, we may denote a random variable and the related deterministic variable/constant as the same symbol since they can be distinguished clearly by context.

For a given portfolio x, the probability of the loss not exceeding a threshold α is given by

$$\Psi(x,\alpha) = \int_{f(x,r)\le\alpha} p(r)dR.$$

Given a confidence level β, the VaR associated with portfolio x is defined as

$$\text{VaR}_\beta(x) = \min\{\alpha \in \Re : \Psi(x,\alpha) \ge \beta\}.$$

The corresponding CVaR is defined as the conditional expectation of the loss of the portfolio exceeding or equal to VaR, i.e.,

$$\text{CVaR}_\beta(x) = \frac{1}{1-\beta}\int_{f(x,r)\ge\text{VaR}_\beta(\text{x})} f(x,r)p(r)dr.$$

Moreover, Rockafellar and Uryasev (2000) prove that CVaR has an equivalent definition as follows:

[7] By way of Rockafellar and Uryasev (2000), all the results can be applied to the case where r follows a discontinuous distribution.

$$\mathrm{CVaR}_\beta(x) = \min_{\alpha \in \Re} F_\beta(x, \alpha),$$

where $F_\beta\ (x, \alpha)$ is expressed as

$$F_\beta(x, \alpha) = \alpha + \frac{1}{1-\beta} \int\limits_{r \in \Re^N} [f(x, r) - \alpha]^+ p(r) dr,$$

where $[\cdot]^+$ is defined as $[t]^+ = \max\{0, t\}$ for any $t \in \Re$. Thus, minimizing CVaR over $x \in X$ is equivalent to minimizing $F_\beta(x, \alpha)$ over $(x, \alpha) \in X \times \Re$, i.e.,

$$\min_{x \in X} \mathrm{CVaR}_\beta(x) = \min_{(x,\alpha) \in X \times \Re} F_\beta(x, \alpha),$$

which implies that a pair (x^*, α^*) solves $\min_{(x,\alpha) \in X \times \Re} F_\beta(x, \alpha)$ if and only if x^* solves $\min_{x \in X} \mathrm{CVaR}_\beta(x)$. Moreover, $F_\beta(x, \alpha)$ is convex with respect to (x, α) and $\mathrm{CVaR}_\beta(x)$ is convex with respect to x; when $f(x, y)$ is convex with respect to x and X is convex, the joint minimization is a convex programming problem, which can be efficiently solved by an interior algorithm.

More specifically, Rockafellar and Uryasev (2000) show that, under the mild assumption of $\beta > 0.5$ and normal distribution with mean $\bar{r}$ and covariance Σ, CVaR reduces to

$$\mathrm{CVaR}_\beta(x) = \kappa_\beta \|\Sigma^{\frac{1}{2}} x\| - \bar{r}' x, \tag{16}$$

where $\kappa_\beta = \frac{-\int_{-\infty}^{\Phi^{-1}(1-\beta)} t\phi(t) dt}{1-\beta}$, $\|\cdot\|$ represents the standard Euclidean norm, $\phi(\cdot)$ and $\Phi(\cdot)$ are the standard normal density and cumulative distribution functions, respectively. In this case, we can write the portfolio optimization problem

$$\min_{x \in X} \ \kappa_\beta \|\Sigma^{\frac{1}{2}} x\| - \bar{r}' x$$

as

$$\min_{(x,\chi) \in X \times \Re^+} \ -\bar{r}' x$$
$$\text{s.t. } \|\Sigma^{\frac{1}{2}} x\| \leq \chi,$$

which has an equivalent counterpart

$$\min_{(x,\chi) \in X \times \Re^+} \ -\bar{r}' x$$
$$\text{s.t. } x' \Sigma x \leq \chi^2.$$

From the convexity of the problem and the Kuhn-Tucker conditions, there exists λ such that the above problem is equivalent to

$$\min_{x \in X} \ \gamma x' \Sigma x - \bar{r}' x,$$

where γ is a function of κ_β. Up to this point, the problem

$$\min\{\mathrm{CVaR}_\beta(x) \ : \ \bar{r}' x \geq r_0, x \in X\}$$

is equivalent to the problem

$$\min\{x^\top \Sigma x \,:\, \bar{r}'x \geq r_0, x \in X\}$$

in the efficient frontier sense when both target constraints are active.

On the other hand, when the distribution is not normal, to deal with the calculation of the integral of the multivariate and non-smooth function in (16), Rockafellar and Uryasev (2000) present an approximation approach via sampling method:

$$F_\beta(x,\alpha) \approx \alpha + \frac{1}{N(1-\beta)} \sum_{k=1}^{N} [f(x, r_{[k]}) - \alpha]^+,$$

where N denotes the number of samples with respect to the portfolio distribution and $r_{[k]}$ denotes the k-th sample (we use the subscript $[k]$ to distinguish a vector from a scalar).

Together with (P_1), the portfolio revision can be cast as

$$\begin{aligned}
(P_3) \qquad \max \quad & (r_f y + \bar{r}'x) - \gamma\left(\alpha + \frac{1}{N(1-\beta)}\sum_{k=1}^{N} u_k\right) \\
\text{s.t.} \quad & u_k \geq -x' r_{[k]} - \alpha, \\
& u_k \geq 0,\ k = 1, \ldots, N, \\
& x = x^0 + x^b - x^s, \\
& y + e'x + \sum_{j=1}^{n} c_j^b x_j^b + \sum_{j=1}^{n} c_j^s x_j^s = 1, \\
& x^b \geq 0,\ x^s \geq 0.
\end{aligned}$$

From a practical perspective, there is a tractable advantage implied in (P_3) that we can solve a mean-variance optimization problem as a linear optimization problem if the portfolio return follows a multivariate normal distribution. This is apparently non-trivial since it eliminates the requirement of estimating the covariance matrix Σ from data, which usually suffers from estimation errors. On the other hand, practical experience suggests that the stability of the numerical solution of linear optimization is stronger than that of quadratic ones in general. In this sense, the optimal portfolio from the mean-CVaR strategy is more robust compared to the mean-variance approach. Notice that for continuous distributions, CVaR has several alternative names such as mean-shortfall or tail VaR (Acerbi and Tasche 2002; Bertsimas et al. 2004).

4.1 Analytical results in the case of one risky asset and one risk-free asset

We consider a portfolio consisting of a risky asset and a risk-free asset with initial endowment x^0 and y^0. These two assets are assumed to have the same characteristics as described in Sect. 3.1: the risky asset has mean $\bar{r}$ and variance σ^2, and the the risk-free asset has constant return rate r_f and variance zero. We also assume the optimal strategy requires buying some risky asset, i.e., $x^b > 0$ and $x^s = 0$.

From the above analysis, the optimal portfolio with mean-CVaR strategy will coincide with the mean-variance optimal portfolio if the risky asset is normally distributed. For this reason, in order to investigate the effect of transaction costs, the risky asset is assumed to have a discrete distribution with sample space $\{r_1, r_2, \ldots, r_N\}$ and $\mathrm{P}\{r = r_j\} = \frac{1}{N}$. Without loss of generality, we also assume that the distribution is asymmetric ($r_j + r_{N\text{-}j} \neq 0$) and $\{r\}_{j=1}^{N}$ is in the increasing order:

$$r_1 \leq r_2 \leq \cdots \leq r_N.$$

Let $S = \lfloor N(1-\beta) \rfloor$ and $\bar{r}_\beta = \frac{1}{S}\sum_{j=1}^{S} r_j$. Using the definition given by (16), we obtain the estimator of CVaR:

$$\mathrm{CVaR}_\beta(x) = -\bar{r}_\beta x. \tag{17}$$

Suppose the optimal portfolio is to buy some risky asset. We write the portfolio optimization problem with transaction costs as

$$\begin{aligned} \max \quad & r_f y + \bar{r}x - \mathrm{CVaR}_\beta(x) \\ \text{s.t.} \quad & x = x^0 + x^b, \\ & y + x + x^b x^b = 1. \end{aligned}$$

This is a simple linear program with only one variable x^b. Keep in mind that there is an implicit constraint, $x^b \geq 0$, since the optimal portfolio is supposed to buy some risky asset. Equivalently, we may rewrite the optimization problem as

$$\max\left\{[\bar{r} + \gamma\bar{r}_\beta - r_f(1+c^b)]x^b : 0 \leq x^b \leq 1 - x^0\right\}. \tag{18}$$

Apparently, the optimal solution is either 0 or $1 - x^0$, which depends on the sign of $\bar{r} + \gamma\bar{r}_\beta - r_f(1+c^b)$. More specifically, if $\bar{r} + \gamma\bar{r}_\beta - r_f(1+c^b) < 0$, we have $x^{b*} = 0$, i.e., there is no rebalancing of the risky asset and the initial portfolio (x^0, y^0) is optimal. On the other hand, if $\bar{r} + \gamma\bar{r}_\beta - r_f(1+c^b) \geq 0$, we have $x^{b*} = 1 - x^0$. One interesting phenomenon is that the optimal portfolio is independent of the variance of the risky asset.

4.2 Analytical results in the case of two risky assets

Now we consider the case where the two assets are risky with mean $\bar{r}_1$ and $\bar{r}_2$ for asset 1 and asset 2, respectively. Following the previous example, the two assets have the discrete sample spaces $\{(r_j^{(1)}, r_j^{(2)})\}_{j=1}^N$ with equal probability support for each scenario. Also, we assume they are independent of each other.

To obtain an analytical solution, suppose $r_1^{(1)} \leq r_2^{(1)} \leq \cdots \leq r_N^{(1)}$ and $r_1^{(2)} \leq r_2^{(2)} \leq \cdots \leq r_N^{(2)}$. Then we have

$$\mathrm{CVaR}_\beta(x) = -\bar{r}_\beta^{(1)} x_1 - \bar{r}_\beta^{(2)} x_2,$$

where $\bar{r}_\beta^{(1)} = \frac{1}{S}\sum_{j=1}^{S} r_j^{(1)}$, $\bar{r}_\beta^{(2)} = \frac{1}{S}\sum_{j=1}^{S} r_j^{(2)}$, and $S = \lfloor N(1-\beta) \rfloor$. Suppose the transaction costs are symmetric and the optimal portfolio requires buying some asset 1 and selling some asset 2 such that $x_1^b > 0$ and $x_2^s > 0$. In addition, we assume $x_1 \leq 1$ and $x_2 \geq 0$.

$$\begin{aligned} \max_{x_1^b, x_2^s} \quad & \bar{r}_1 x_1 + \bar{r}_2 x_2 - \gamma(-\bar{r}_\beta^{(1)} x_1 - \bar{r}_\beta^{(2)} x_2) \\ \text{s.t.} \quad & x_1^b = x_1 - x_1^0,\ x_2^s = x_2^0 - x_2, \\ & x_1 + x_2 + x_1^b c^b + x_2^s c^s = 1. \end{aligned} \tag{19}$$

Rearranging problem (19), we have

$$\max\left\{\left[\bar{r}_1 + \gamma\bar{r}_\beta^{(1)} - \lambda(1+c^b)\right]x_1 - [\bar{r}_2 + \gamma\bar{r}_\beta^{(2)} - \lambda(1-c^s)]x_2^s : x_1^0 \leq x_1 \leq 1, 0 \leq x_2 \leq x_2^0\right\}.$$

Again, to satisfy our assumption of buying asset 1 and selling asset 2, we have

$$\bar{r}_1 + \gamma \bar{r}_\beta^{(1)} - \lambda(1 + c^b) > 0, \bar{r}_2 + \gamma \bar{r}_\beta^{(2)} - \lambda(1 - c^s) < 0.$$

As a result, the optimal solution is

$$x_1^b = 1 - x_1^0, x_2^s = x_2^0.$$

5 Extension

In practice, the mean return and the covariance matrix are typically estimated based on a sample of historical returns. These estimated values typically have large errors, particularly for the mean returns. The mean-variance model can be very sensitive to the estimation error in mean return: small differences in the estimate of $\bar{r}$ can result in large variations in the optimal portfolio composition. To protect the performance against estimation risk and to alleviate the sensitivity of the mean-variance model to uncertain input estimates, min-max robust optimization can be employed to generate the portfolio which has the best performance under the worst-case scenarios. As demonstrated by Broadie (1993), because the covariance matrix can typically be estimated more accurately than the mean return, in this paper we ignore robust consideration with respect to Σ.

We first consider the min–max robust mean-variance portfolio:

$$\begin{aligned} \min_{x \in X} \quad & \max_{r} -r'x + \gamma x'\Sigma x \\ \text{s.t.} \quad & (r - \bar{r})'\Sigma^{-1}(r - \bar{r}) \le \chi, \end{aligned} \tag{20}$$

where γ is the risk-aversion parameter and χ is a non-negative number.

For any feasible $x \in X$, let us look at the following optimization problem with respect to r:

$$\begin{aligned} \min_{r \in \Re^N} \quad & r'x \\ \text{s.t.} \quad & (r - \bar{r})\Sigma^{-1}(r - \bar{r}) \le \chi^2. \end{aligned}$$

Apparently, the objective is linear and the constraint is quadratic. Consequently, we can use the necessary and sufficient Kuhn-Tucker condition to obtain its optimal solution

$$r^* = \bar{r} - \frac{\chi}{\sqrt{x'\Sigma x}}$$

with unique optimal objective value

$$\bar{r}'x - \chi\sqrt{x'\Sigma x}.$$

Therefore, problem (20) reduces to the following second-order cone program:

$$\min_{x \in X} -\bar{r}'x + \chi\sqrt{x'\Sigma x} + \gamma x'\Sigma x.$$

Since this is a convex programming problem, it is easy to show that there exist $\tilde{\chi} \ge 0$ such that the above problem is equivalent to

$$\begin{aligned} \min_{x \in X} \quad & -\bar{r}'x + \gamma x'\Sigma x \\ \text{s.t.} \quad & \sqrt{x'\Sigma x} \le \tilde{\chi}, \end{aligned}$$

which is further equivalent to

$$\begin{aligned} \min_{x \in X} \quad & -\bar{r}'x + \gamma x'\Sigma x \\ \text{s.t.} \quad & x'\Sigma x \leq \tilde{\chi}^2. \end{aligned}$$

According to the convexity of the problem and the Kuhn-Tucker conditions, there exists a $\hat{\gamma} \geq 0$ such that the above problem is equivalent to

$$\min_{x \in X} -\bar{r}'x + \tilde{\gamma} x'\Sigma x,$$

where $\tilde{\gamma} = \gamma + \hat{\gamma}$.[8] This is a surprising result. The robust mean-variance approach cannot improve the portfolio performance, and instead, it has the same efficient frontiers as the classical mean-variance model.

Let us revisit the traditional mean-variance model. Given the optimal portfolio is very sensitive to the return mean, here we consider a CVaR robust mean-variance optimization by replacing the mean by a CVaR measure (Zhu et al. 2007), that is

$$\min_{x \in X} \mathrm{CVaR}_\beta(x) + \gamma x'\Sigma x. \tag{21}$$

Problem (21) can be explained in this way. In the Markowitz framework, the investor's optimal strategy is to simultaneously maximize the expected return, $\bar{r}'x$, and minimize the variance, $x'\Sigma x$. On the other hand, we can also explain the negative of $\bar{r}'x$ as the expected loss of the portfolio. That means that theoretically the investor hopes to optimize the portfolio by minimizing both the loss and variance. In practice, however, a plethora of empirical studies (see, for example, Black and Litterman 1992; Broadie 1993) show that a small perturbation of the inputs may lead to a large change in the optimal portfolio, implying that the parameters should be estimated as precisely as possible. To reduce or eliminate the estimation error in the mean, we may adopt an alternative by minimizing the expected return of the worst-case scenarios, i.e., minimizing CVaR. In other words, while $\min -\bar{r}'x$ minimizes the expectation of all the losses, $\min \mathrm{CVaR}_\beta(-\bar{r}'x)$ only minimizes the expectation of those losses larger than VaR.

Now we are in a position to present the CVaR-variance portfolio revision problem with transaction costs as the following optimization problem:

$$\begin{aligned} & \min \alpha + \frac{1}{N(1-\beta)} \sum_{k=1}^{N} u_k + \gamma x'\Sigma x \\ & \text{s.t. } u_k \geq -x^\top r_{[k]} - \alpha, \\ & \quad u_k \geq 0, \; k = 1, \ldots, N, \\ & \quad x = x^0 + x^b - x^s, \\ & \quad y + e'x + \sum_{j=1}^{n} c_j^b x_j^b + \sum_{j=1}^{n} c_j^s x_j^s \leq 1, \\ & \quad x^b \geq 0, \; x^s \geq 0. \end{aligned}$$

It should be mentioned that our CVaR-variance model is different from the mean-variance-CVaR model recently developed by Roman et al. (2007). While their model provides an improved solution when a mean-variance efficient portfolio has an excessively large CVaR or a mean-CVaR efficient portfolio has an excessively large variance, our model principally deals with the difficulty of estimating the expected return that has been

[8] For a detailed proof, see Zhu et al. (2007).

addressed by several studies. Similarly, while we can also consider the situation where transaction costs are paid at the end of the planning period, we omit it here.

6 Practical application

In this section, we present an illustrative example to demonstrate the impact of transaction costs on the optimal portfolio to be rebalanced. In particular, we consider the problem of portfolio revision using 10 industry portfolios with equal initial endowment. As mentioned in Sect. 2, we normalize the initial portfolio, hence $x_j^0 = 0.1$ for $j = 1, \ldots, 10$. The data are available from the website of Kenneth French.[9] Daily average value-weighted returns are considered from January 1, 2000 to December 31, 2007.

First, we consider the impact of transaction costs on the efficient frontiers of mean-variance and mean-CVaR optimal portfolios. In particular, the optimal portfolios can be obtained by solving problems (P_2) and (P_3) respectively. From Fig. 1, transaction costs must obviously be taken into account when employing an active portfolio trading strategy. To highlight the effect, we consider three situations with different transaction costs, $c_j^b = c_j^s = 0$, $c_j^b = c_j^s = 1\ \%$, and $c_j^b = c_j^s = 2.5\ \%$.[10] As can be seen in the figure, the efficient frontier is dramatically lowered by the transaction costs in a nonlinear pattern.

Here, we compare the performance of the mean-variance and mean-CVaR strategies with and without transaction costs. As we mentioned earlier, as well as in Rockafellar and Uryasev (2000), for normal distribution, or even the ellipsoidal distribution, these two methodologies are equivalent in the sense that they generate the same efficient frontier. In practice, however, for the non-normal, and non-symmetric distribution in general, mean-variance and mean-CVaR may generate totally different optimal portfolios. While the mean-variance model penalizes both the loss and gain deviating from the expected return, the mean-CVaR approach aims at the left tail of the portfolio returns, corresponding to the high losses.

Without considering the transaction costs, Fig. 2 displays the mean-variance (left panel) and mean-variance (right panel) efficient portfolios in both the variance-expected return space and the CVaR-expected return space. In this section, we set all the confidence levels β at 0.95. Notice in Fig. 2 that although the efficient frontiers are very close to each other, the discrepancy between mean-variance and mean-CVaR solutions does exist. To explore this discrepancy, we compare both solutions with respect to the ability to control downside risk when revising the portfolio. More specifically, for the case where the expected return is 0.1050, Fig. 3 shows the left tails of the empirical distributions of returns associated with the mean-variance and mean-CVaR optimal portfolios.[11] Apparently, the left tails of the return distribution of the mean-CVaR optimal portfolio are thinner than that of the mean-variance portfolio, implying the mean-CVaR optimal portfolio can perform better relative to the mean-variance methodology in the case of downside world (other results are available upon request). Figures 4 and 5 display efficient frontiers in the presence of

9 http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data-library.htm.

10 We acknowledge that the transaction cost of 2.5 % may be a little bit large in practice and we use it here only to make the figure look apparent.

11 To make the difference apparent, we only show the left tail/part of the whole distribution of portfolio return. This figure is only used to show that mean-CVaR can generate a portfolio with a thinner tail relative to mean-variance. In this sense, it has nothing to do with with extreme value theory that focuses on tail distribution and has attracted a lot of attention in risk management and insurance.

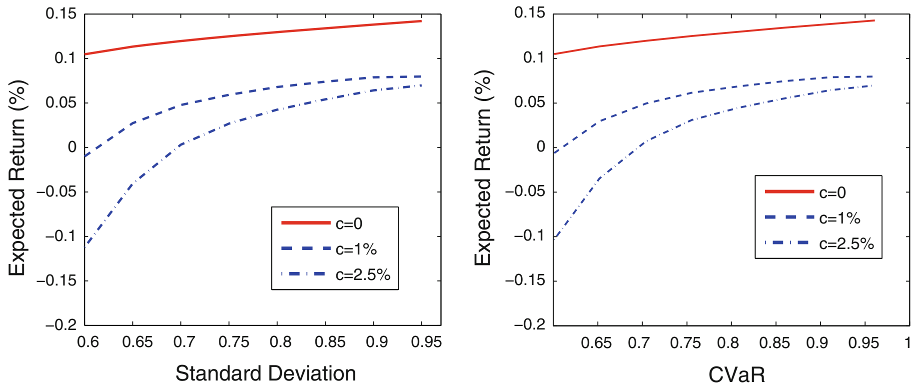

Fig. 1 Efficient frontier of optimal portfolios with mean-variance (*left panel*) and mean-CVaR (*right panel*) strategies in the presence of symmetric transaction costs $c_j^b = c_j^s = 0$, 1, and 2.5 %

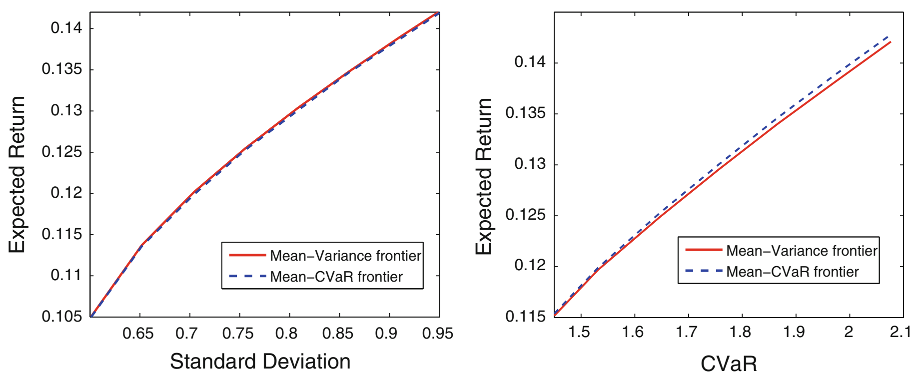

Fig. 2 Efficient frontiers of mean-variance and mean-CVaR optimal portfolios in the standard deviation-expected return (*left panel*) and the CVaR-expected return spaces (*right panel*) without transaction costs

transaction costs. The discrepancy between the mean-variance and mean-CVaR solutions are marginally significant relative to the case where transaction costs are ignored.

In general, the difference between the mean-variance and the mean-CVaR approaches is not very significant. However, the close efficient frontiers do not imply that the mean-variance optimal portfolio is close to the mean-CVaR optimal portfolio, and vice versa. In their study, Krokhmal et al. (2001) suggest that the closeness of efficient portfolios generated from the CVaR and mean-variance optimizations may be attributable to their specific dataset where the returns did not depart significantly from the normal distribution. On the other hand, the risk-expected return efficient frontier may be misleading for daily returns. While a difference of 0.01 is marginal for one day in Figs. 4 and 5, it may yield a 2.5 discrepancy for one year (250 trading days).

7 Conclusion

In the portfolio revision process, there are always transaction costs associated with buying and selling an asset due to brokerage fees, bid-ask spreads, taxes, etc. In this paper, we

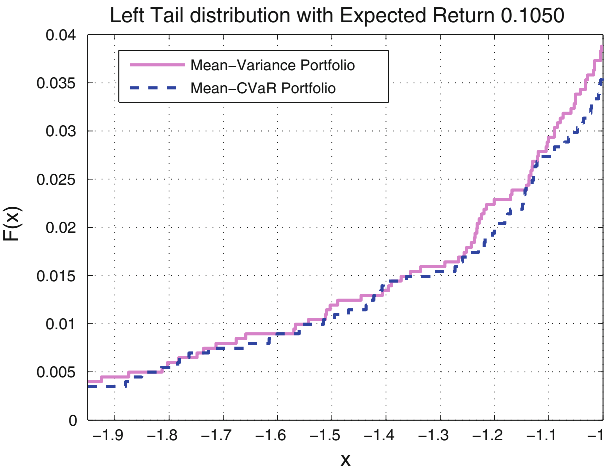

Fig. 3 Left tail of empirical distribution

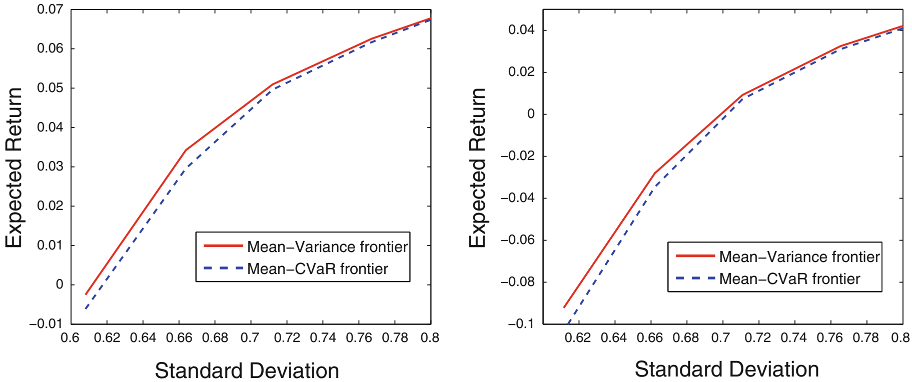

Fig. 4 Efficient frontiers of mean-variance and mean-CVaR optimal portfolios in the standard deviation-expected return space with transaction costs $c_j^b = c_j^s = 1\ \%$ (*left panel*) and $c_j^b = c_j^s = 2.5\ \%$ (*right panel*)

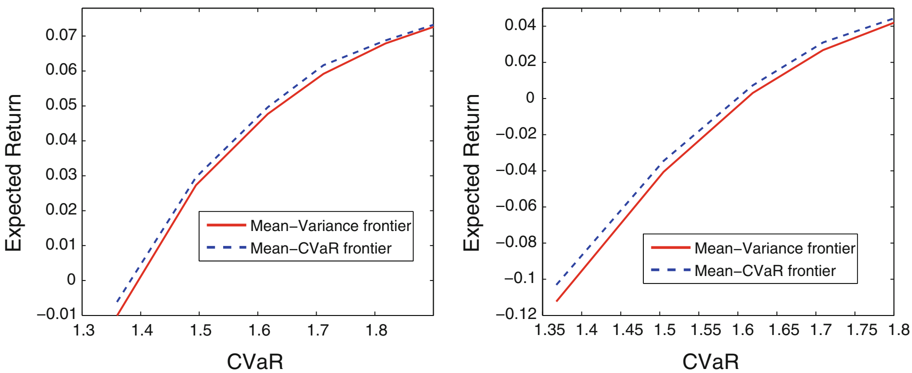

Fig. 5 Efficient frontiers of mean-variance and mean-CVaR optimal portfolios in the CVaR-expected return space with transaction costs $c_j^b = c_j^s = 1\ \%$ (*left panel*) and $c_j^b = c_j^s = 2.5\ \%$ (*right panel*)

consider the problem of portfolio revision with transaction costs which are paid at the beginning of the planning horizon. We demonstrate that the impact of transaction costs can be integrated into both the classical mean-variance and mean-CVaR frameworks and that even some analytical solutions under mild assumptions can be obtained via optimization techniques.

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